Multiple removal based on wavefield extrapolation

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1 Introduction

Either forward or inverse modeling is required to deal with the properties of the subsurface that determine the reflected waves. Inverse modeling becomes very complicated, particularly when there are problems of instability and non-uniqueness. Forward modeling is simpler, but even so it becomes complicated for a general situation. In this section I will present the basic theory and simple examples concerning forward and inverse modeling following closely the work of Berkhout (1985). At the end the theory of multiple attenuation based on wavefield extrapolation will be considered.

The data represent upgoing reflected waves, related to downgoing source waves. Hence, a wave separation of the measured seismic data must be previously applied. This method is called one way approach because the downgoing and upgoing fields are computed separately. Another approach is the two way technique in that the total response (primaries and multiples) is computed by extrapolating simultaneously the continuous wavefield components P and $\frac{1}{\rho}\frac{\partial P}{\partial z}$.

2 Theory

Wavefield extrapolation is based on the Kirchhoff integral, which comes from substituting the wave equation into Green's second theorem

$$\int_V \left[F\nabla^2 G - G\nabla^2 F\right] dV = \oint_S \left[F\nabla G - G\nabla F\right].\hat{\mathbf{n}}\ dS. \tag{1}$$

Suppose we have a closed, source free surface S. We choose F as the pressure field which is generated by sources outside S

$$F = P(x, y, z, \omega) \tag{2}$$

where P satisfies

$$\nabla^2 P + k^2 P = 0. \tag{3}$$

G is chosen as the Fourier Transformed pressure for a compressional wavefield which is generated by a monopole in a point A inside S (Green's function)

$$G = \frac{\exp(-jkr)}{r} \tag{4}$$

with

$$r = \sqrt{(x-x_a)^2 + (y-y_a)^2 + (z-z_a)^2}. \tag{5}$$

G satisfies

$$\nabla^2 G + k^2 G = -4\pi\delta(x-x_a)\delta(y-y_a)\delta(z-z_a) \tag{6}$$

and Green's theorem becomes

$$\oint_S [P\nabla G - G\nabla P] \,.\, \vec{n}\, dS = -4\pi \int_V P\delta(x-x_a)\delta(y-y_a)\delta(z-z_a)dV \tag{7}$$

or using the shift property

$$\oint_S [P\nabla G - G\nabla P] \,.\hat{\mathbf{n}}\; dS = -4\pi P_A. \tag{8}$$

where P_A is the wavefield in x_a, y_a. The motion equation states the relation between acceleration and the pressure gradient

$$\frac{\partial P}{\partial n} = -\rho_0 \frac{\partial V_n}{\partial t}. \tag{9}$$

As usual we consider $V(x,y,z,t) = f(x,y,z)\exp(-j\omega t)$

$$\frac{\partial P}{\partial n} = \rho_0 (j\omega\rho_0 V_n). \tag{10}$$

By substituting

$$\oint_S \left[P\frac{\partial G}{\partial n} + (j\omega\rho_0 V_n) G \right] .\hat{\mathbf{n}}\; dS = -4\pi P_A \tag{11}$$

Substituting G, and $\frac{\partial G}{\partial n}$

$$P_A = \frac{1}{4\pi}\oint_S \left[P\frac{\partial\left(\frac{\exp(-jkr)}{r}\right)}{\partial n} + (j\omega\rho_0 V_n)\frac{\exp(-jkr)}{r} \right] .\hat{\mathbf{n}}\; dS \tag{12}$$

This expression tell us how to compute P_A at any point in a source free halfspace due to sources in the other halfspace. Using the Green's function in Eq. (4), the normal derivative results in

$$\frac{\partial G}{\partial n} = \frac{1+jkr}{r^2}\exp(-jkr)\cos\phi \tag{13}$$

$$\cos\phi = \frac{\partial r}{\partial n} \tag{14}$$

Equation (12) is the Kirchhoff's integral for a homogeneous medium. This integral integral states that any pressure field may be synthesized by means of a monopole and a dipole distribution on a closed surface S. The strength of each monopole is given by the normal component of the velocity in S, the strength of each dipole is given by the pressure in S.

Kirchhoff's integral is not very useful in practice because we need to know the pressure and particle velocity data on a closed surface. However, from the Kirchoff integral it is possible to derive the Rayleigh integrals, which are very useful for seismic applications.

Let us choose for closed surface S the plane $z = 0$ and a hemisphere in the top half space. Letting the radius of the hemisphere becomes infinitely large, we can reduce the surface integral to the plane $z = 0$. Eq. (4) for Green's function was a particular function chosen to satisfy the Green's second theorem but it also could be chosen with a constant H and still satisfy Green's theorem

$$G = \frac{\exp(-jkr)}{r} + H. \tag{15}$$

If H is such that

$$\frac{\partial G}{\partial n} = \frac{1 + jkr}{r^2} \exp(-jkr) \cos\phi = 0 \tag{16}$$

on $z = 0$, the same previous derivation for P_A produces the so called Rayleigh integral of type I:

$$P_A = \frac{j\omega\rho_0}{2\pi} \int_{L_x} \int_{Ly} V_n \frac{\exp(-jkr)}{r} dxdy \tag{17}$$

and it represents the pressure in S due to a monopole source on $z = 0$.

By a similar argument we can choose H such that $G = 0$ for $z = 0$. This produces the Rayleigh integral of the second kind. The Rayleigh II integral states that any pressure field can be synthesized by a dipole distribution on the plane $z = 0$.

$$P(x_A, y_A, z_{i-1}, \omega) = \frac{1}{2\pi} \int_{Lx} \int_{Ly} P(x, y, z_i, \omega) \frac{1 + jkr}{r^2} \exp(-jkr) \cos\phi dxdy \tag{18}$$

where $\cos\phi = \frac{\partial r}{\partial n} = \frac{z}{r}$

or its 2D version

$$P(x_A, z_{i-1}, \omega) = -\frac{jk}{2} \int_{Lx} P(x, z_i, \omega) H_1^{(2)}(k\Delta r) \cos\phi dx \tag{19}$$

where $H_1^{(2)}$ is the first order Hankel function of the second class

$$H_1^{(2)} = \frac{1}{j\pi} \int_{-\infty}^{0} \frac{\exp(z/2)(t - 1/t)}{t^2} dt \tag{20}$$

To illustrate the wavefield extrapolation principles I will use the 2D version of the Rayleigh integral of type II, i.e., equation (19). Wavefield extrapolation can be performed in space-time, space-frequency or wavenumber-frequency. I will describe here space-frequency and wavenumber-frequency approaches only.

Defining

$$W(x_A - x, \Delta z, \omega) = -\frac{jk}{2} \cos\phi H_1^{(2)}(kr) \tag{21}$$

we can write

$$P(x_A, z_{i-1}, \omega) = \frac{1}{2\pi} \int_{Lx} W(x_A - x, \Delta z, \omega) P(x, z_i, \omega) dx \tag{22}$$

or

$$P(x, z_{i-1}, \omega) = W(x, \Delta z, \omega) * P(x, z_i, \omega) \tag{23}$$

For the 3D case

$$W(x_A - x, y_A - y, \Delta z, \omega) = \frac{jk}{2\pi} \left[\frac{1 + jkr}{jkr} \cos\phi \right] \frac{\exp(-jkr)}{r}. \tag{24}$$

and

$$P(x, y, z_{i-1}, \omega) = W(x, y, \Delta z, \omega) * P(x, y, z_i, \omega) \tag{25}$$

Hence forward extrapolation in the space-frequency domain can be formulated in terms of convolution along the spatial axes, x and y. If we consider a dipole at $z = z_i$, then the response at depth level $z = z_{i-1}$ is given by $W(x_r - x_s, y_r - y_s, \Delta z, \omega)$, so that W is called the spatial impulse response or the spatial wavelet for the temporal frequency ω. If the velocity varies laterally, the spatial wavelet $W(x_r - x_s, y_r - y_s, \Delta z, \omega)$ becomes space-variant, that is $W(x_s, x_r - x_s, y_r - y_s, \Delta z, \omega)$, and the convolution is valid only if an average value of the velocity can be used within the operator length. Generally, for space-variant discrete situations and a finite operator length, a matrix formulation is used.

If we consider the situation without lateral variations in thickness or velocity, then the function W does not change along the spatial coordinates x and y so that the convolution can be carried out by means of a multiplication in the ω-k_x domain

$$\widetilde{P}_{i-1}(k_x, k_y, z_{i-1}, \omega) = \widetilde{W}(k_x, k_y, \Delta z, \omega)\widetilde{P}_i(k_x, k_y, z_i, \omega) \tag{26}$$

where

$$\widetilde{W}(k_x, k_y, \Delta z, \omega) = \mathcal{F}[\frac{j|z|}{2\pi}(\frac{1+jkr}{jkr^3})\exp(-jkr)] = \exp(-j\sqrt{\frac{\omega^2}{c^2} - (k_x^2 + k_y^2)}\ \Delta z) \tag{27}$$

$$\widetilde{W}(k_x, k_y, \Delta z, \omega) = \exp(-jk_z\Delta z) \tag{28}$$

F stands for Fourier transform and k_z is the vertical component of the wavenumber.

This result can be also obtained from the Helmholtz equation. Substituting a solution like

$$p = p(x, y, z)\exp(-i\omega t) \tag{29}$$

into the wave equation we obtain

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0. \tag{30}$$

Fourier transformation with respect to x and y produces the one-dimensional Helmholtz equation;

$$\frac{\partial^2 P}{\partial z^2} + (k^2 - k_x^2 - k_y^2)P = 0 \tag{31}$$

whose solution is

$$P(k_x, k_y, z, \omega) = A(k_x, k_y, \omega)\exp(\pm jk_z|z - z_i|) \tag{32}$$

By taking $z \longrightarrow z_i$ the integration constant $A_i = P(k_x, k_y, z_i, \omega)$ and

$$P(k_x, k_y, z, \omega) = P(k_x, k_y, z_i, \omega)\exp(\pm jk_z|z - z_i|) \tag{33}$$

With the propagator $W(x, y, z_i, z_{i-1}, \omega)$ it is possible to perform the forward extrapolation of P from z_{i-1} to z_i. This operator can be applied recursively to go from any z_i to any z_j

where $z_j > z_i$. In the same way an inverse extrapolator can be defined to go from z_i to z_{i-1} or working recursively from any z_j to z_i where $z_j < z_i$.

Now, if we have a downward propagating source $S^+(x,y,\omega)$ at the surface and a series of reflectivities $R(x,y,z_m,\omega)$,with $m = 1, N$, the wavefield $P(x,y,z_0,t)$ can be obtained as

$$P(x,y,z_0,t) = \mathcal{F}^{-1}[D(z_{0,}\omega)*[\sum_{m=1}^{N} W^-(z_0,z_m,\omega)*R(z_m,\omega)*W^+(z_m,z_0,\omega)]*S^+(z_0,\omega)] \quad (34)$$

where $\mathcal{F}^{-1}[.]$ is the inverse Fourier Transform, and the notation has been simplified making implicit that all terms are x,y dependent, $D(x,y,z_{0,}\omega)$ accounts for the properties of the receivers and, in the simplest case $D(x,y,z_{0,}\omega) = I$.

If we reverse one of the operators, set it as block tridiagonal matrix and set the other operator as columns of a second matrix, the convolutions can be carried out by means of matrix multiplication. To do this a matrix $P(\omega)$ is built up for each frequency setting the shot gathers as columns. In this way the rows contain the receiver gathers. To visualize this, consider a 2D data set with different shots, each one with its corresponding shot gather. Setting every shot gather behind the previous one, a 3 dimensional matrix is constructed as $\mathbf{P}(t,x_r,x_s)$ where t is the time, x_r represents the coordinates of the receivers, and x_s the coordinates of the sources. Fourier transforming produces $\mathbf{P}(\omega,x_r,x_s)$. An index permutation gives $\mathbf{P}(x_r,x_s,\omega)$. With these matrices and the shifted operator $\mathbf{W}(x,z_0,z_m\omega)$ set as

$$\begin{matrix} \overrightarrow{\mathbf{W}}(-x,z_0,z_m\omega_i) & \overrightarrow{0} & \cdots & \overrightarrow{0} \\ 0 & \overrightarrow{\mathbf{W}}(-x+1,z_0,z_m\omega_i) & \cdots & \overrightarrow{0} \\ \cdots & \cdots & \cdots & \cdots \\ \overrightarrow{0} & \overrightarrow{0} & .. & \overrightarrow{\mathbf{W}}(-x+n,z_0,z_m\omega_i) \end{matrix} \quad (35)$$

where all the elements are vectors, the convolution can be performed as a matrix multiplication,

$$\overrightarrow{\mathbf{W}}(x,z_0,z_m,\omega_i) * \overrightarrow{\mathbf{P}}(x,z_m,\omega_i) = \mathbf{W}(x,z_0,z_m,\omega_i)\overrightarrow{\mathbf{P}}^{\mathbf{T}}(x,z_m,\omega_i). \quad (36)$$

Hence it is possible to compute the response of the earth, frequency by frequency, with all the sources.

$$\mathbf{P}(z_0,\omega) = \mathbf{D}(z_{0,}\omega)[\sum_{m=1}^{N} \mathbf{W}^-(z_0,z_m,\omega)\mathbf{R}(z_m,\omega)\mathbf{W}^+(z_m,z_0,\omega)]\mathbf{S}^+(z_0,\omega). \quad (37)$$

Here all the terms are x,y dependent and

$$\mathbf{W}^+(z_m,z_0,\omega) = \mathbf{W}^+(x,y,z_m,z_0,\omega)$$

is the downward propagator from z_0 to z_m,

$$\mathbf{W}^-(z_m, z_0, \omega) = \mathbf{W}^-(x, y, z_m, z_0, \omega)$$

is the upward propagator from z_m to z_0,

$$\mathbf{P}(z_i, \omega) = \mathbf{P}(x, y, z_i, \omega)$$

is the wavefield at $z = z_i$. In the simplest 1D case (no x, y variation in the earth properties) with sources along x it is possible to work in the $k_x - \omega$ domain

$$P(z_0, k_x, \omega) = D(z_0, k_x, \omega).[\sum_m W^-(z_0, z_m, k_x, \omega) R(z_m, k_x, \omega) W^+(z_m, z_0, k_x, \omega)] S(z_0, k_x, \omega) \tag{38}$$

where all matrix multiplications have been replaced by scalar multiplication.

Let us simplify the problem taking $\mathbf{D} = \mathbf{I}$, that is one receiver at every x position and calling

$$\mathbf{T}(z_0) = \sum_m \mathbf{W}^-(x, y, z_0, z_m, \omega)\mathbf{R}(x, y, z_m, \omega)\mathbf{W}^+(x, y, z_m, z_0, \omega), \tag{39}$$

$\mathbf{T}(z_0)$ is an operator which takes the downward wavefield P^+ at level z_0, downward propagates it until $z = z_m$, computes the reflected upward wavefield P^- at $z = z_m$, and propagates it upward until $z = z_0$ and finally adds all the wave fields coming from all layers m. In the absence of multiples the wavefield at $z = z_0$ will be

$$\mathbf{P}^-(z_0) = \mathbf{T}(z_0)\mathbf{S}^+ \tag{40}$$

where $\mathbf{S}^+$ is the downward field from the source.

Multiples can be included in this formulation because these are generated by feed back of previously generated waves. An expression which includes surface related multiples, i.e., all multiple reflections that have been reflected at least once from the free surface can be obtained as follows. If there is a surface with reflection coefficient r_0 the upward wavefield $\mathbf{P}^-(z_0)$ will produce a downward field

$$\mathbf{P}^+(z_0) = -r_0\mathbf{P}^-(z_0). \tag{41}$$

The total downward field will be

$$\mathbf{P}^+_{\mathbf{inc}}(z_0) = \mathbf{S}^+ - r_0\mathbf{P}^-(z_0) \tag{42}$$

and the response of the earth will be

$$\mathbf{P}^-(z_0) = \mathbf{T}(z_0)\mathbf{P}^+_{\mathbf{inc}}(z_0) \tag{43}$$

or

$$\mathbf{P}^-(z_0) = \mathbf{T}(z_0)[\mathbf{S}^+ - r_0\mathbf{P}^-(z_0)] \tag{44}$$

which is a recursive filter, whose present output depends on past outputs.

$$\mathbf{P}^-(z_0) + r_0{}^+\mathbf{T}(z_0)\mathbf{P}^-(z_0) = \mathbf{T}(z_0)\mathbf{S}^+ \tag{45}$$

Solving for $\mathbf{P}^-(z_0)$ the formulation

$$\mathbf{P}^-(z_0) = [\mathbf{I} + r_\mathbf{0}\mathbf{T}(z_0)]^{-1}\mathbf{T}(z_0)\mathbf{S}^+ \tag{46}$$

generates all the free surface multiples. Expanding

$$[\mathbf{I} + r_\mathbf{0}\mathbf{T}(z_0)]^{-1} = \sum_{m=0}^{\infty}(-r_\mathbf{0})^\mathbf{n}\mathbf{T}^\mathbf{n}(z_0) \tag{47}$$

$$\mathbf{P}^-(z_0) = \mathbf{P}^-(z_0) + \sum_{m=1}^{\infty}(-r_0)^n\mathbf{T}^\mathbf{n}(z_0)\mathbf{S} \tag{48}$$

where taking more terms produces higher order surface multiples. To include all possible multiples, free surface as well as internal, we have to extend the recursive primary modeling scheme further by adding, during each upward continuation step, all multiples related to the current surface. For example, if we have arrived at the level z_{m+1} with an upward travelling reflected wavefield $\mathbf{P}^-(z_{m+1})$ which includes all multiples, then

$$\mathbf{P}^-(z_m) = \mathbf{W}^-(z_m, z_{m+1})[\mathbf{R}^+(z_{m+1}) + \mathbf{P}^-(z_{m+1})]\mathbf{W}^+(z_{m+1}, z_m) \tag{49}$$

where $\mathbf{P}^-(z_m)$ represents the total upward travelling response from depth level $z > z_m$ assuming zero reflectivity at $z = z_m$. To include all multiples associated with $z = z_m$ we need another step to include the feedback of $\mathbf{P}^-(z_m)\ \mathbf{R}^-(z_m)$.

$$\mathbf{P}^-_\mathbf{tot}(z_m) = [\mathbf{I} - \mathbf{P}^-(\mathbf{z_m}).\mathbf{R}^-(z_m)]^{-1}\mathbf{P}^-(z_m) \tag{50}$$

or to reduce instability,

$$\mathbf{P}^-_\mathbf{tot}(z_m) = \mathbf{P}^-(z_m) + \sum_{n=1}^{\infty}[\mathbf{R}^-(\mathbf{z_m}).\mathbf{P}^-(z_m)]^n\mathbf{P}^-(z_m) \tag{51}$$

where only some of the infinity terms are calculated. In conclusion, we start at maximum depth and continue up to the surface according to the previous expressions (49) and (50). When we arrive at the surface, we have created the total response, i.e., primaries and all possible multiple reflections. In the last step the source matrix and the detector matrix can be included.

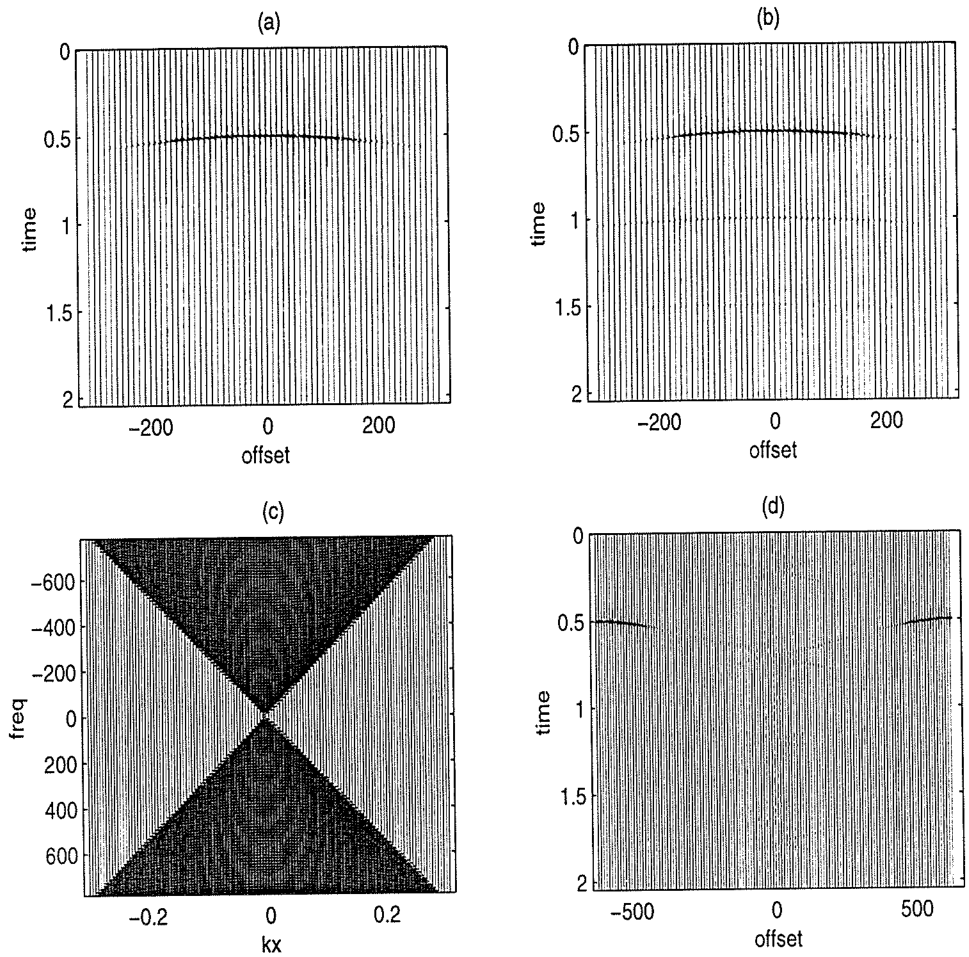

Figure 1: Forward modeling in ω-k_x domain: One layer and halfspace. (a) Primary, (b) primary+multiple, (c) propagator $k_x - \omega$, (d) propagator $x - t$.

2.1 Example: single source. Forward modeling in k_x-ω

Figure 1 and 2 show an example of forward modeling in the ω-k_x domain, with a source located at the centre of the line and a simple reflectivity independent of frequency and incident angle. A simple two-layer and halfspace 1_D model is computed. Figure 1a shows the first step, the primary reflected at surface 2. The next step, Figure 1b, is to add the multiples produced by the reflection from surface 1. Figure 1c shows the propagator $W(k_x, \omega)$, and Figure 1d presents $W(x,t)$. Evanescent fields have been avoided setting zero the amplitude of the propagator for angles greater than critical, i.e. those with $k_x > k$.

Another consideration regarding the obliquity factor k_z must be implemented. Given an offset interval dx the Nyquist spatial wavenumber $k_N = \frac{\pi}{dx}$. This means that to avoid

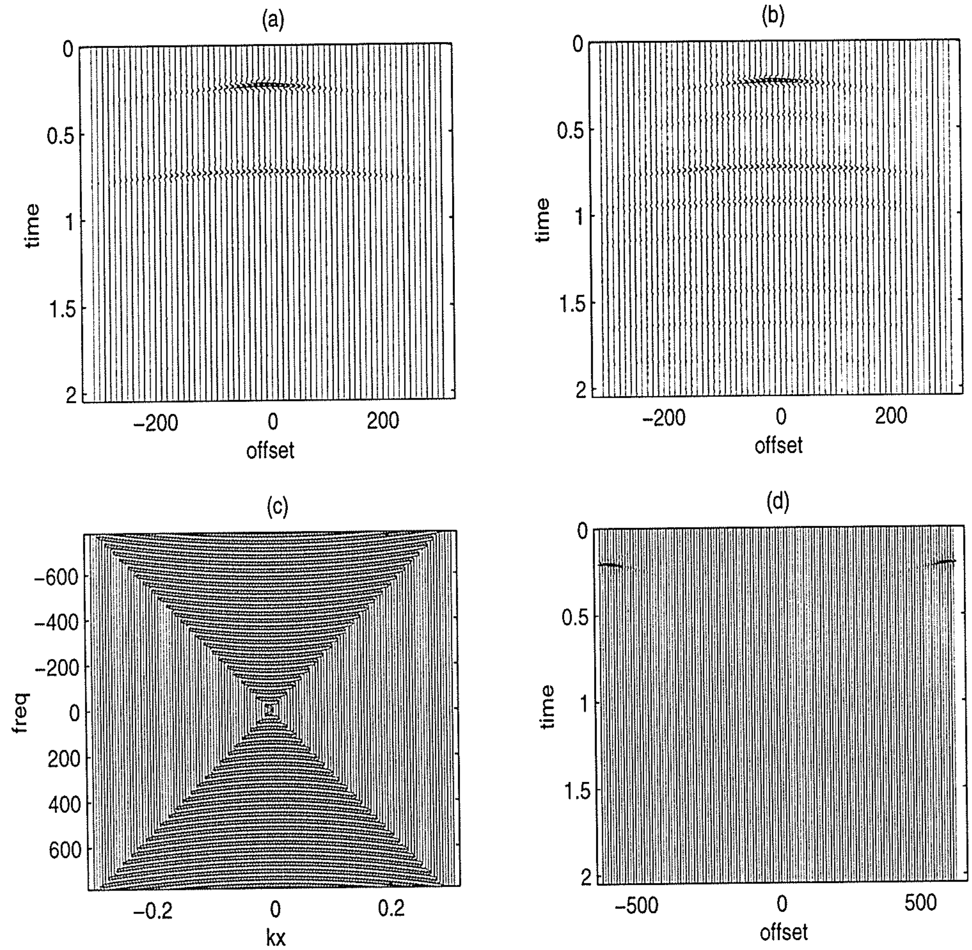

Figure 2: Forward modeling in ω-k_x domain: One layer and halfspace. (a) Primary, (b) primary+multiple, (c) propagator $k_x - \omega$, (d) propagator $x - t$.

aliasing the maximum incidence angle must be $k_x = k \sin\phi < k_N$ or

$$\frac{\omega}{c}\sin\phi < \frac{\pi}{dx} \tag{52}$$

For $\omega_N = \frac{2\pi}{2dt}$, to avoid aliasing the maximum angle is given by

$$\sin\phi_{\max} < \frac{dt}{dx}c. \tag{53}$$

On the plane $k_x - \omega$, all points (wave fronts) with

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2} < k\cos\phi_{\max}. \tag{54}$$

must be zeroed.

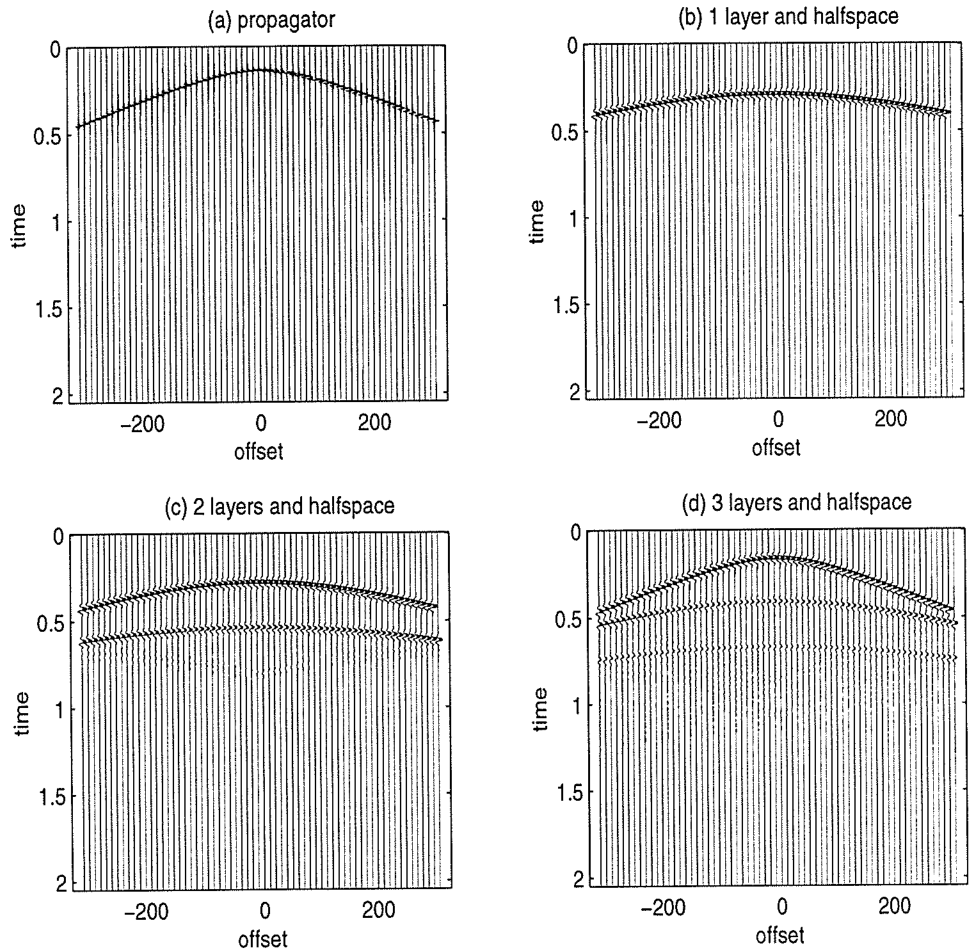

Figure 3: Forward modeling in ω-x domain: Three layers and halfspace. (a) propagator $(x - t)$ for the upper layer, (b) primary from layer 3, (c) primary from layers 3 and 2, (d) primaries from layer 3, 2 and 1.

For the Nyquist temporal frequency $\omega = 2\pi f = \frac{\pi}{dt}$ this condition is satisfied automatically for the maximum k_x on the plane, but for lower frequencies a filter is applied.

Figure 2 shows the same plots for the second step, i.e., after adding layer 1. Figure 2a shows the primary reflected at surface 1 plus primary and multiples from below (internal multiples). In Figure 2b the multiples produced by the reflection on the free surface have been added. (c) shows the propagator $W(k_x, \omega)$, and Figure 2d $W(x, t)$. If more layers are present the process proceeds in the same manner until the surface.

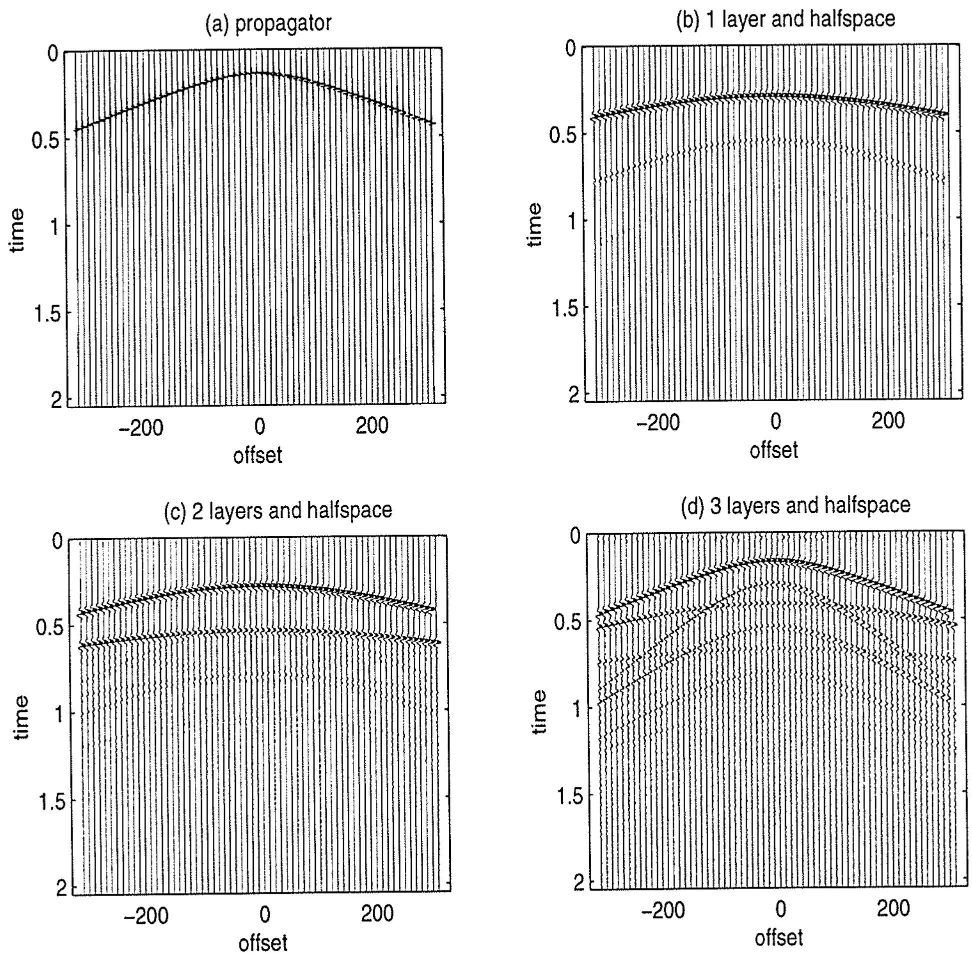

Figure 4: Forward modeling with multiples in ω-x domain: Three layers and halfspace. (a) propagator $(x - t)$ for the upper layer, (b) layer 3, (c) layers 3 and 2, (d) layers 3, 2 and 1.

2.2 Example: Single source. Forward modeling in x-ω

Figure 3 shows forward modeling in the x-ω domain without multiples, and Figure 4 with multiples. The propagator is defined as

$$W(x_A - x, \Delta z, \omega) = -\frac{jk}{2} \cos \phi H_1^{(2)}(kr) \tag{55}$$

and the simplified version for $kr >> 1$ (far field condition) is used

$$W(x_A - x, \Delta z, \omega) = \sqrt{\frac{jk}{2\pi}} \cos \phi \frac{\exp(-ikr)}{r} \tag{56}$$

Figure 3a shows the propagator for the upper layer in the $x - t$ domain, and it represents the wavefield in a halfspace at level (x, z) when the impulsive source is at (x_A, z_o). When we add

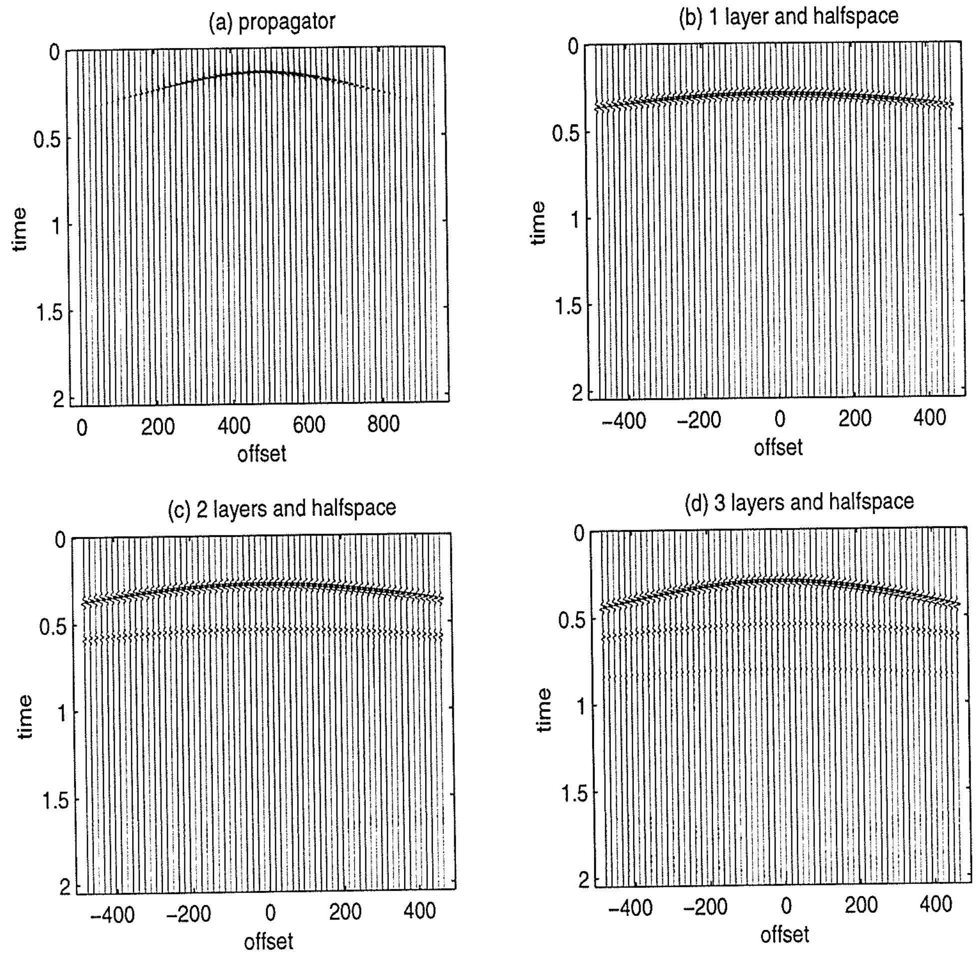

Figure 5: Forward modeling for multiple sources in ω-x domain: three layers and halfspace. (a) Propagator $(x-t)$ for the upper layer, (b) layer 3 ,(c) layers 3 and 2 (d)layer 3,2 and 1.

a layer to the halfspace, the primary wavefield consists of the primary reflection between the layer and halfspace Figure 3b but the total wavefield has the primary and all its multiples (Figure 4b).

Figure 3c-d show the primaries with 2 and 3 layers respectively. In Figure 4c-d the multiples have been added. The multiples are stronger for the surface because the surface reflection coefficient is equal to -1.

2.3 Example: Multiple sources. Forward modeling in x-ω

Here there is an example with a formulation for many simultaneous sources. Only one source per shot is considered but all the shot gathers are computed simultaneously and the

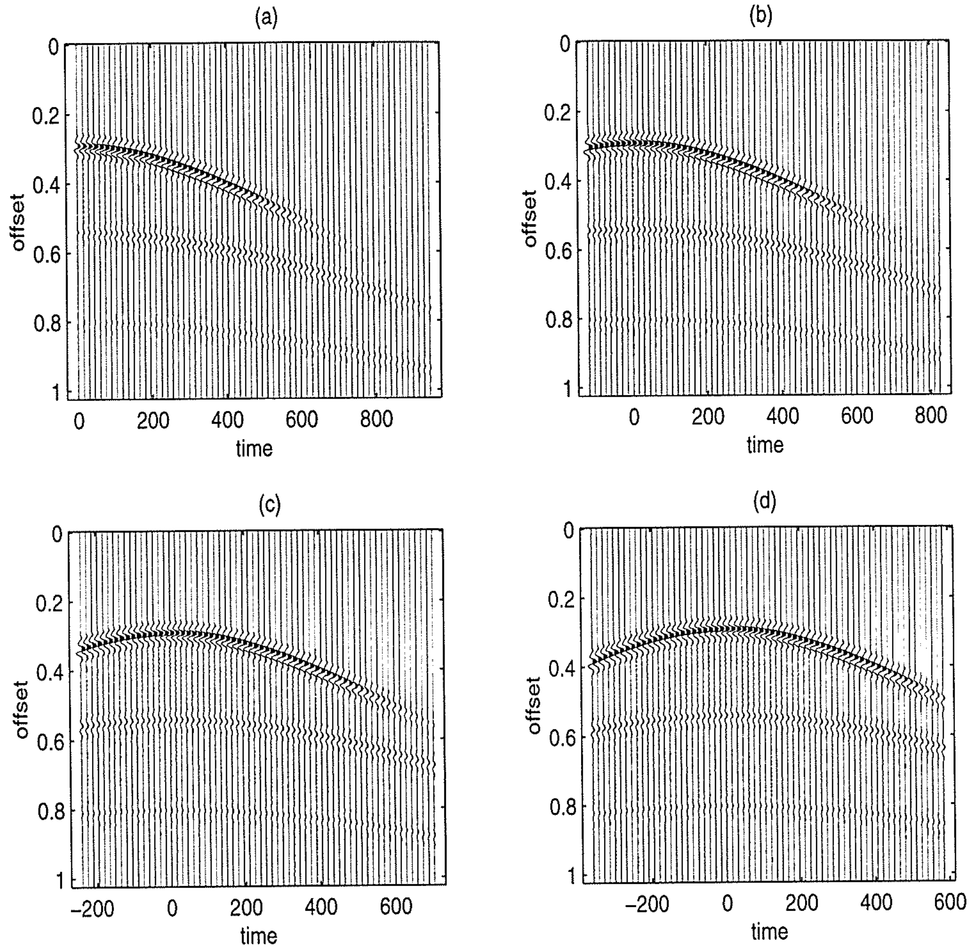

Figure 6: Source gathers without multiples: three layers and halfspace. Four different source positions.

formulation is able to handle the case of simultaneous sources. Using matrix multiplication to perform spatial convolution, a full set of shot records and detector gathers is computed, resulting in a 3_D matrix with $x = x_d, y = x_s$, and $z = \omega$. Shot gathers are stored as rows and detector gathers as columns (frequencies are in the third dimension). First an example with only primaries is shown in Figure 5 and 6, then the same model with the inclusion of first and second order primaries is shown in Figure 7 and 8. Figure 5a shows the propagator $W(x,t)$ for the upper layer, Figure 5b presents a shot gather for a shot at the centre of the line, and Figure 5c-d show the same for two and three layers respectively. Figure 6a-c show four shot gathers for different shots from the left to the right side of the line. Figure 7 and 8 are the same but with first and second order multiples added as is explained below.

The forward problem is implemented as follows. A set of matrices is defined for the sources,

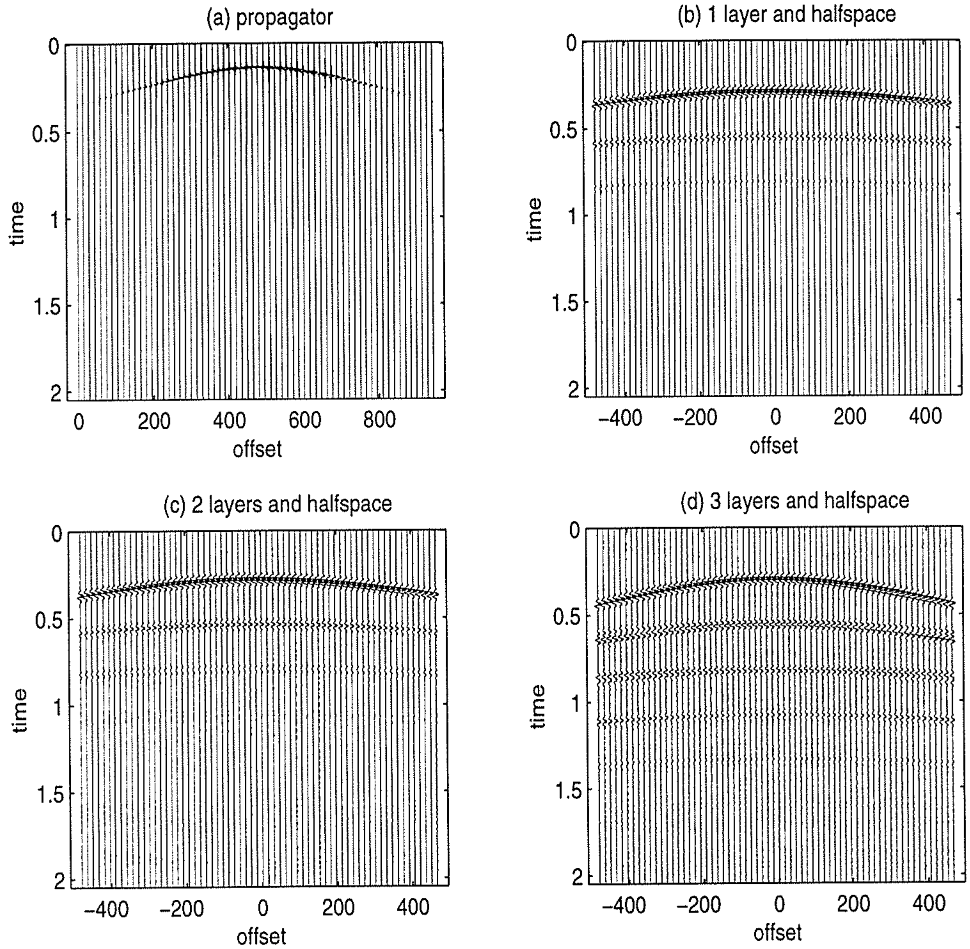

Figure 7: Forward modeling with multiples for multiple sources in ω-x domain: three layers and halfspace. (a) Propagator $(x-t)$, (b) layer 3 ,(c) layers 3 and 2 (d)layer 3,2 and 1.

the reflectivity and the propagator

$$\mathbf{S}(\omega, x_s) = FFT(\mathbf{S}(t, x_s)) \tag{57}$$

$$\mathbf{R_m}(x_r, x_s) \cong Rm_{i,i}(x_r, x_s) \tag{58}$$

$$W(x_A - x, \Delta z, \omega) = \sqrt{\frac{jk}{2\pi}} \cos\phi \frac{\exp(-ikr)}{r} \tag{59}$$

where x_r are receiver positions and x_s are source positions. The wavefield is computed for every frequency ω_i. Hence, a loop is required for frequencies and another for layers. The wavefield is given by

$$\mathbf{P}^-(z_m) = \mathbf{W}^-(z_m, z_{m+1})[\mathbf{R}^+(z_{m+1}) + \mathbf{P}^-_{\mathbf{tot}}(z_{m+1})].\mathbf{W}^+(z_{m+1}, z_m) \tag{60}$$

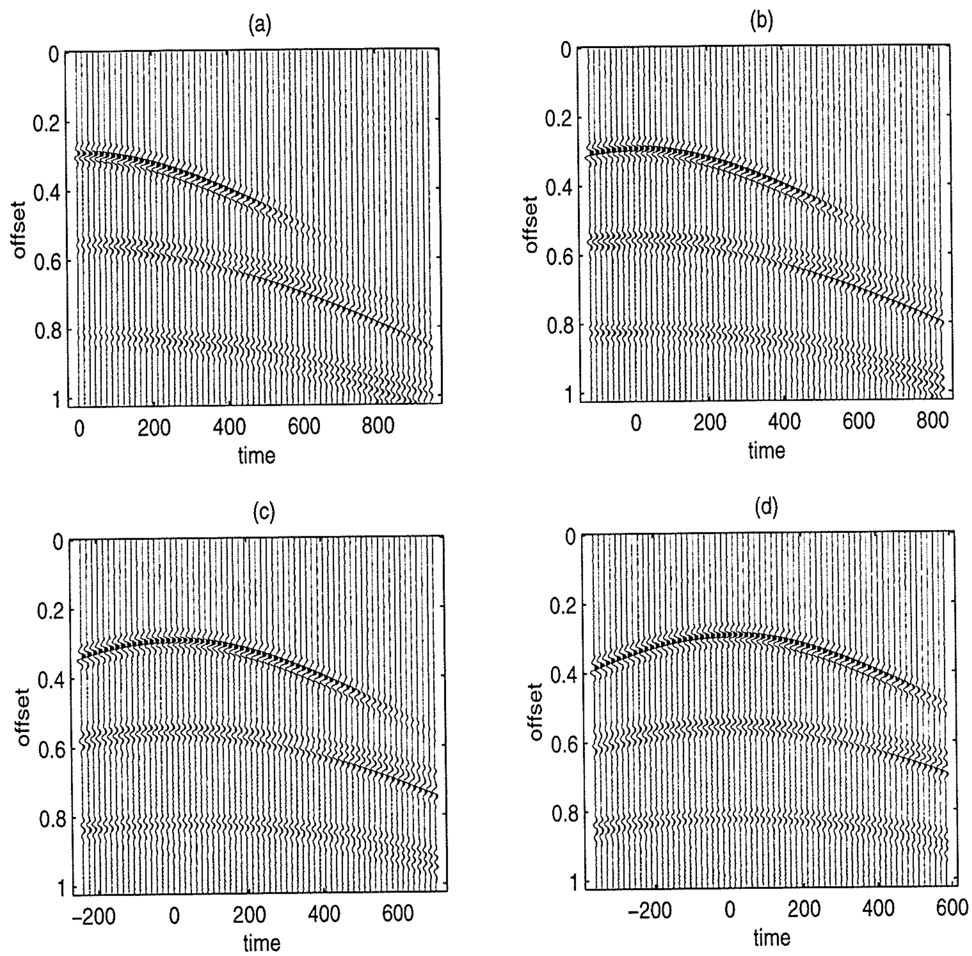

Figure 8: Source gathers with multiples: three layers and halfspace. Four different source positions.

and the multiples are included with

$$\mathbf{P}^{-}_{\mathbf{tot}}(z_m) = [\mathbf{I} - \mathbf{R}^{-}(\mathbf{z_m}).\mathbf{P}^{-}(z_m)]^{-1}\mathbf{P}(z_m) \tag{61}$$

Because of the inherent instability of the process of deconvolution, special care must be taken at this step. A band limited version of the multiples is evaluated, with only those frequencies that yield a reasonable condition number for the matrix $[\mathbf{I} - \mathbf{R}^{-}(\mathbf{z_m}).\mathbf{P}^{-}(z_m)]$. Also the diagonal of the matrix is increased to decrease the condition number, a procedure that decreases the amplitude of the multiples. Because of this instability I have calculated only first and second order multiples. Given the finite extension of the record only a very small number of multiples will appear.

The wavefield results in a 3D matrix with $x = x_s, y = x_d$ and $z = \omega$. Considering one frequency ω, every row represents a shot gather and every column represents a detector gather. To obtain any source or detector record an index permutation is done

$$\mathbf{P}(\omega, x_r, x_s) = Permute(\mathbf{P}(x_r, x_s, \omega)). \tag{62}$$

The wavefield is multiplied by the source in the frequency domain and the inverse Fourier transform is applied to obtain the synthetic data,

$$\mathbf{P}(t, x_r, x_s) = \mathcal{F}^{-1}\mathbf{P}(\omega, x_r, x_s).\mathbf{S}(\omega, x_s). \tag{63}$$

3 Inversion

Inversion of real seismic data is a very complicated task, because of the under determinated nature of the problem and inherent instability. The waves propagate inside the earth and each contrast in the physical properties of the medium produces a complicate pattern of primaries and multiples. Information about all physical parameters of the earth that affect the waves is in the data, but to extract this information we need to know the details of the physical phenomenon. In theory, given the seismic response at the surface, information about density and velocity in the subsurface could be computed. Geophysicist try to develop the information and the most probable result is presented in terms of zero-offset reflectivity. An important step for performing a complete inversion is to understand that multiples are part of the seismic experiment. Usually, they are considered coherent noise and as such, they are attenuated or eliminated as much as possible. Hence, inverse modeling requires three steps:

1- Inverse wavefield extrapolation.

2- Reflectivity estimation.

3- Multiple removal.

Inverse wavefield extrapolator: The propagation effects in a layer can be quantified by spatial convolution for each temporal frequency component with an operator W:

Downward continuation:

$$P^+(x, y, z_m, \omega) = W^+(x, y, \Delta z_m, \omega) * P^+(x, y, z_{m-1}, \omega) \tag{64}$$

Upward continuation:

$$P^-(x, y, z_{m-1}, \omega) = W^-(x, y, \Delta z_m, \omega) * P^-(x, y, z_m, \omega) \tag{65}$$

To compensate the propagation effects in layer (z_m, z_{m-1}) we need an operator for each frequency such that

Downward continuation:

$$\left\langle P^+(x,y,z_{m-1},\omega)\right\rangle = F^+(x,y,\Delta z_m,\omega) * P^+(x,y,z_m,\omega) \tag{66}$$

Upward continuation:

$$\left\langle P^-(x,y,z_m,\omega)\right\rangle = F^-(x,y,\Delta z_m,\omega) * P^-(x,y,z_{m-1},\omega) \tag{67}$$

$F^\pm(x,y,\Delta z_m,\omega)$ are called inverse wavefield extrapolators. Application involves a deconvolution process along the spatial axes for each frequency component. F and W are inverse operators of each other

$$W^\pm(x,y,\Delta z_m,\omega) * F^\mp(x,y,\Delta z_m,\omega) = \delta(x)\delta(y) \tag{68}$$

In situations with lateral variations a matrix formulation is necessary,

$$\mathbf{W}^+(z_m,z_{m-1}) * \mathbf{F}^-(z_{m-1},z_m) = \mathbf{I}, \tag{69}$$

$$\mathbf{F}^+(z_m,z_{m-1}) * \mathbf{W}^-(z_{m-1},z_m) = \mathbf{I}, \tag{70}$$

In the ω-k_x domain the relation is expressed as

$$W^\pm(k_x,k_y,\Delta z,\omega)F^\mp(k_x,k_y,\Delta z,\omega) = 1. \tag{71}$$

A simple calculation of F would be,

$$F^\mp(k_x,k_y,\Delta z,\omega) = 1/W^\pm(k_x,k_y,\Delta z,\omega) \tag{72}$$

However this calculation is unstable, because it defines an exponentially increasing operator which is unacceptable in practical situations.

Several alternatives are possible:

Band limited version:

$$\widetilde{F} = \widetilde{A}/\widetilde{W} \tag{73}$$

where $\widetilde{A}$ represents a spatial lowpass amplitude weighting function. This could be determined for example by the maximum dip angle.

Least squares inversion

$$\widetilde{F} = \frac{\widetilde{W}^*}{|\widetilde{N}|^2 + |\widetilde{W}|^2} \tag{74}$$

similar to the two sided least squares temporal deconvolution.

Matched filter

$$\widetilde{F} = \widetilde{W}^* \tag{75}$$

Reflectivity: As was explained before the forward model for the prestack data is obtained as

$$\mathbf{P}^-(z_0) = \mathbf{D}(z_0).[\sum_{\mathbf{m}} \mathbf{W}^-(z_0, z_m)\mathbf{R}(z_m).\mathbf{W}^+(z_m, z_0)].\mathbf{P}^+(z_0) \tag{76}$$

Hence, the response for a single depth level $z = z_m$ can be written as

$$\mathbf{P}^-_{\mathbf{m}}(z_0) = \mathbf{W}^-(z_0, z_m)\mathbf{R}(z_m).\mathbf{W}^+(z_m, z_0) \tag{77}$$

To obtain information about the reflectivity $\mathbf{R}(z_m)$ the response from a single layer must be inverted:

$$\mathbf{R}^-(z_m) = \mathbf{F}^+(z_m, z_0)\mathbf{P}^-_{\mathbf{m}}(z_0).\mathbf{F}^-(z_0, z_m) \tag{78}$$

Removing all propagation effects between depth levels z_0 and z_m with inverse wavefield extrapolation, the events reflected at the depth level $z = z_m$ are extrapolated to the surface, i. e., $t = 0$. Hence after inverse wave field extrapolation to depth level z_m the data at $t = 0$ are integrated on frequency to obtain the reflectivity,

$$\mathbf{R}_{\mathbf{m}}(z_0) = \frac{1}{\pi} Re \int_{\omega_{\min}}^{\omega_{\max}} [\mathbf{F}^+(z_0, z_m)\mathbf{P}(z_m)\mathbf{F}^-(z_m, z_0)] d\omega \tag{79}$$

Multiple Removal To perform the inversion, it is necessary to remove the multiples from the wavefield $\mathbf{P}(z_m)$ every time we eliminate one layer from the surface wavefield. This process may be unstable because it involves deconvolution of a band limited wavefield.

It was stated before that the multiples are generated by a feed back process

$$\mathbf{P_{tot}}(z_m) = [\mathbf{I} - \mathbf{R}(\mathbf{z_m})\mathbf{P}(z_m)]^{-1}\mathbf{P}(z_m) \tag{80}$$

then

$$\mathbf{P}(z_m) = [\mathbf{I} + \mathbf{R}(\mathbf{z_m})\mathbf{P_{tot}}(z_m)]^{-1}\mathbf{P_{tot}}(z_m) \tag{81}$$

Thus, starting with the surface data, it is possible to remove all the surface related multiples. Because the reflection coefficient for the surface is approximately equal to -1 these multiples are the strongest in the data. To remove internal multiples we perform inverse wavefield extrapolation, estimate the reflectivity and then, as before, we can remove all the multiples associated with that surface.

A computation of $[\mathbf{I} - \mathbf{R}(\mathbf{z_m}).\mathbf{P_{tot}}(z_m)]^{-1}$ results in instability if strong multiples are present (Verschuur et al., 1992). To understand this, the inverse can be developed as a series

$$\mathbf{P}(z_m) = \mathbf{P_{tot}}(z_m) - \sum_{n=1}^{\infty}(-1)^{n-1}[\mathbf{P}^{-}_{\mathbf{tot}}(z_m)\mathbf{R}(\mathbf{z_m})]^n\mathbf{P}^{-}_{\mathbf{tot}}(z_m) \tag{82}$$

The inverse implies an infinite number of terms. In the presence of strong multiple reflections, the series expansion converges very slowly, and straightforward inversion is unstable. Taking only a limited number of terms into account stabilizes the inversion. The number of terms that should be taken into account depends on the highest order surface related multiples present in the data because each additional term taken into account results in eliminating surface related multiples of one order higher.

In Berkhout and Verschuur (1997), the IIR filter from equation (82), is performed as a recursive filter

$$\mathbf{P^{(n)}}(z_m) = \mathbf{P_{tot}}(z_m) - \mathbf{F^{(n)}}(z_m)\mathbf{P_{tot}}(z_m) \tag{83}$$

$$\mathbf{F_{tot}}(z_m) = \mathbf{P^{(n-1)}}(z_m)\mathbf{R}(z_m) \tag{84}$$

$$\mathbf{P^{(0)}}(z_m) = \mathbf{P_{tot}}(z_m) \tag{85}$$

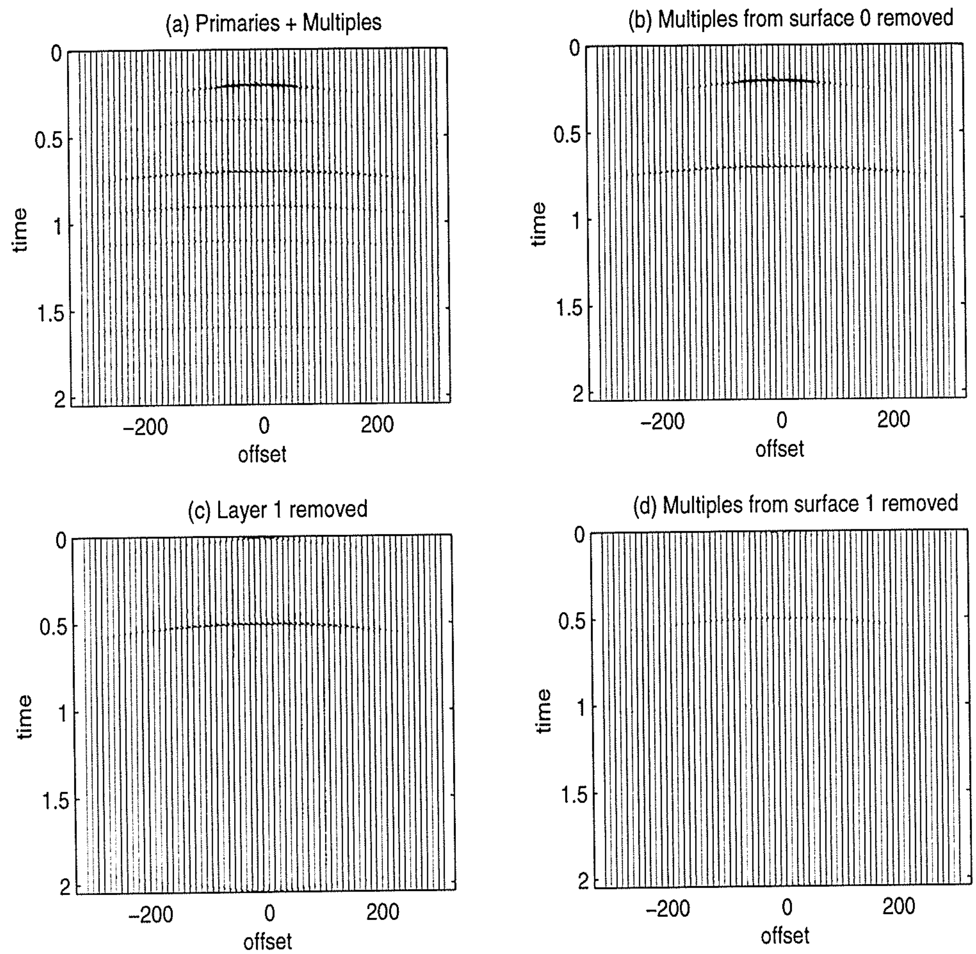

Figure 9: Inverse modeling in ω-x domain: two layers. (a) Full wavefield, (b) surface multiples removed, (c) layer 1 removed, (d) internal multiples removed.

3.1 Example: inversion

In Figure 9 the basic steps of inverse wavefield extrapolation are presented. Figure 9a presents the initial total wavefield (primaries and multiples) obtained with a forward modeling. Figure 9b shows the wavefield after removal of the free surface. This operation has removed all free-surface-related multiples and only primaries from surface 1 and 2 and internal multiples are present. The following step is the downward continuation of $P(z_0)$ to $P(z_1)$, (figure9-c) that is the wavefield after removing the first layer, remaining only the second primary and internal multiples. This step moves the primary reflection at surface 1 to the surface at $z = 0$. The reflection coefficient here could be obtained by imaging but to keep the example simple I have taken the values used in the forward model instead. Then, the reflection coefficient is subtracted and the wavefield is ready for the next step, the removal of internal multiples reflected at the surface 1. Finally, figure 9d must be the wavefield with all internal multiples removed and only the second primary must be present (a multiple still remains because the inversion was not perfect). The last step is the downward continuation of the wavefield from $z = z_1$ until $z = z_2$, obtaining $P(z_2)$ and imaging to get the reflection coefficient at surface 2.

3.2 Surface multiple attenuation

According Verschuur (1992), the historical development of the method of multiple attenuation starts with Anstey and Newman (1967) who observed that by means of the autoconvolution of a trace, primary events were transformed into multiples. Riley and Claerbout (1976) used this idea in the so called Noah's deconvolution. Kennet (1979) described an inversion scheme in the ω-k_x domain. Berkhout (1982) redefined the multiple problem for laterally varying media by using a wave theory-based matrix formulation. Iterative versions of multiple attenuation are in Verschuur et. al. (1989, 1992) and Wapenaar et al. (1990). Another approach for attenuating free surface and internal multiples is based on a point scatterer model (see Weglein et al., 1997), but this approach will not be discuss here.

Even when inversion can in theory eliminate all the multiples as part of the inversion itself, the process is very complicated so that a common approach is to predict the multiples applying some iterative method, starting from some approximation of the multiple free data. The predicted multiples are subtracted from the initial data to attenuate the multiples.

Surface multiple attenuation (Dragoset and Jericevic, 1998) is based on the concept that every surface multiple consists of segments that, from a surface perspective, are primary events. The surface multiple attenuation algorithm manipulates and combines the primary events in a seismic data set so as to predict the surface multiple wavefield and then uses that prediction to cancel the actual surface multiple. Because the method does not depend on move out discrimination it does not affect primaries. Multiples can be considered primary reflections whose sources are other primary events (Jakubowicz, 1998). Hence, a multiple (response of the earth system) can be obtained as the convolution of a primary (source) with another primary (impulse response of the earth system).

The task of combining primary events to predict multiples (Dragoset and Jericevic, 1998) is similar to the diffraction aperture problem of classical optics and, as such, can be solved by means of the Kirchhoff integral. Multiples can be predicted according to the Kirchhoff integral with the following formula:

$$m(s,r,t) = -\sqrt{t}F_{\omega\to t}\left\{(1-i)\sqrt{\frac{\omega}{4\pi}}\int_A dx F_{t\to\omega}[\sqrt{t}p_s(x,t)] \right.$$
$$\left. F_{k_x\to x}\left(\sqrt{1-(k_xV/\omega)^2}F_{x\to k_x}\left(F_{t\to\omega}[\sqrt{t}p_r(x,t)]\right)\right)\right\} \quad (86)$$

Where F stands for Fourier transformation, with the change of variable indicated by the subscript and V is the wave velocity. A is the surface aperture and is equivalent to a cable length. The operation implied by this formula is as follows. For every x location in the aperture, temporally convolve a trace from a common shot record $p_s(x,t)$ with the corresponding trace from a common receiver record $p_r(x,t)$; then stack the convolution result and multiply by -1 (to account for the surface reflection coefficient). The integral over the variable x accomplishes the stack, while the convolution is accomplished by multiplication in the frequency domain. Various other pieces of the equation are simply to ensure that the operation on 2-D data produces a result that correctly honors the physics of 3-D sound wave propagation from point sources. The result of equation 86, $m(s,r,t)$ is a single trace that contains a predicted suite of multiples. The whole procedure can be thought of as a 2-D generalization of the idea of predicting multiples by convolution. To predict all of the first order surface multiple wavefield, equation 1 must be computed once for each trace in the data set using

$$M_1 = PO_kP. \quad (87)$$

Here, M represents the entire first order surface multiple wavefield, P represents the primary wavefield, and O_k represents the Kirchhoff operation.

Higher order multiples M_i are obtained as (see Dragoset and Jericevic, 1998)

$$M_i = PO_kM_{i-1}. \quad (88)$$

Thus, if P is known any desired order of multiples can be generated using the same scheme. The entire surface-multiple wavefield M is given by $M_1 + M_2 + ... + Mn$, where n is the highest order surface multiple that can appear in a given data set. The entire recorded wavefield consists of

$$D = P + M_1 + M_2 + ... + M_n \quad (89)$$

or using the recursive formulation for the multiples

$$D = P(1 + O_k(D - M_n)). \tag{90}$$

If it is possible to compute the inverse of the bracketed expression then P can be obtained from the data

$$P = D(1 + O_k(D - M_n))^{-1}. \tag{91}$$

To compensate for the source wavelet contained in predicted multiples, we have to perform a convolution with the inverse of the wavelet w, written as w^{-1}

$$P = D(1 + w^{-1} * O_k(D - M_n))^{-1}. \tag{92}$$

To implement Equation 86, the convolution is performed by means of matrix multiplication, using the following definitions

$$m(s,r,t) = \sqrt{t}F_{\omega\to t}\left\{m(s,r,\omega)\right\} \tag{93}$$

$$m(s,r,\omega) = -\sum_x p_s(x,\omega)p_r(x,\omega) \tag{94}$$

$$p_s(x,\omega) = F_{t\to\omega}[\sqrt{t}p_s(x,t)] \tag{95}$$

$$p_r(x,t) = (1-i)dx\sqrt{\frac{\omega}{4\pi}}F_{k_x\to x}\left\{\sqrt{1-\left(\frac{k_xV}{\omega}\right)^2}F_{x\to k_x}\left\{F_{t\to\omega}[\sqrt{t}p_r(x,t)]\right\}\right\} \tag{96}$$

For each particular ω, s and r, the product of matrices $p_s(x,\omega)$ and $p_r(x,\omega)$ produces one element in the matrix $m(s,r,\omega)$.

The physical meaning of the Kirchhoff's integral is the following. The primary source located at a particular posicion produces waves that, after reflecting at the subsurface, reachs the surface and become secondary sources. First order surface multiples are primaries originated for these secondary sources. Hence, they can be calculated by adding the temporal convolution of sources and impulse response functions of the earth for every surface location. Let us consider a shot gather in the $x-\omega$ domain, corresponding to a shot at position x_{sp}. In our formulation, a shot gather for ω_i when the source is at x_{sp} is a row of the three dimensional matrix storing the wavefield $\mathbf{P}(\mathbf{x_s}, \mathbf{x_r}, \omega_\mathbf{i})$. Every element of this row vector contains the secondary source at position x_{rp}, when the source is at x_{sp}. A receiver gather at x_{rm} is given by one column of the matrix $\mathbf{P}(\mathbf{x_s}, \mathbf{x_r}, \omega_\mathbf{i})$. Every element of this column vector, $P(x_{rp}, x_{rm}, \omega_i)$ contains the transfer function of the earth at the position x_{rm} due to a source located at x_{rp}. Hence, the frequency ω_i of the first order multiple at position x_{rm}, when the source is at x_{sp} is given by

$$m_1(x_{sp}, x_{rm}, \omega_i) = \sum_{rp} P(x_{sp}, x_{rp}, \omega_i) P(x_{rp}, x_{rm}, \omega_i) \tag{97}$$

Note that the calculation depends on destructive and constructive interference to produce the correct response. This explain why boundary effects appears at the end of the synthetic traces and also the problem due to missed traces.

3.3 Example

A very simple example is presented in fig 10. Figure 10a shows a single synthetic shot record with two primaries, one at $0.27s$ (at zero offset), and another at $0.77s$. The forward model was computed in the x-ω domain from a two layer and halfspace model. Applying equation (86) to the data (all shot and receiver gathers included) produces all first order surface multiples. The matrix multiplication of the data with themselves (with the definitions given above to honor the wave equation) reproduces the first order multiple for every shot and detector. There are some boundary effects, produced by the finite aperture of the model. The waves generated by the secondary sources at the end of the gathers are incompletely cancelled at one side by another secondary source, but not by the other side.

Figure 10b displays the first order multiple. The first event at $0.54s$. is the multiple travelling twice in the upper layer. At $1.03s$. appears a multiple that has travelled twice in the upper layer and once in the second layer. At $1.54s$. the wave that has travelled twice all the way from the surface to the second interface has arrived. No other surface multiple exist, but a new computation of the product will produce the second order multiples.

Figure 11a presents one synthetic shot gather from a two layer model, upper layer with a velocity of $2000m/s$ and thickness of $200m$, second layer with velocity of $3000m/s$ and thickness of $350m$. The primaries are at 0.2 s and 0.5 sec. First order surface multiples and first and second order internal multiples are in the data as follows.

P_1 at $0.2s$

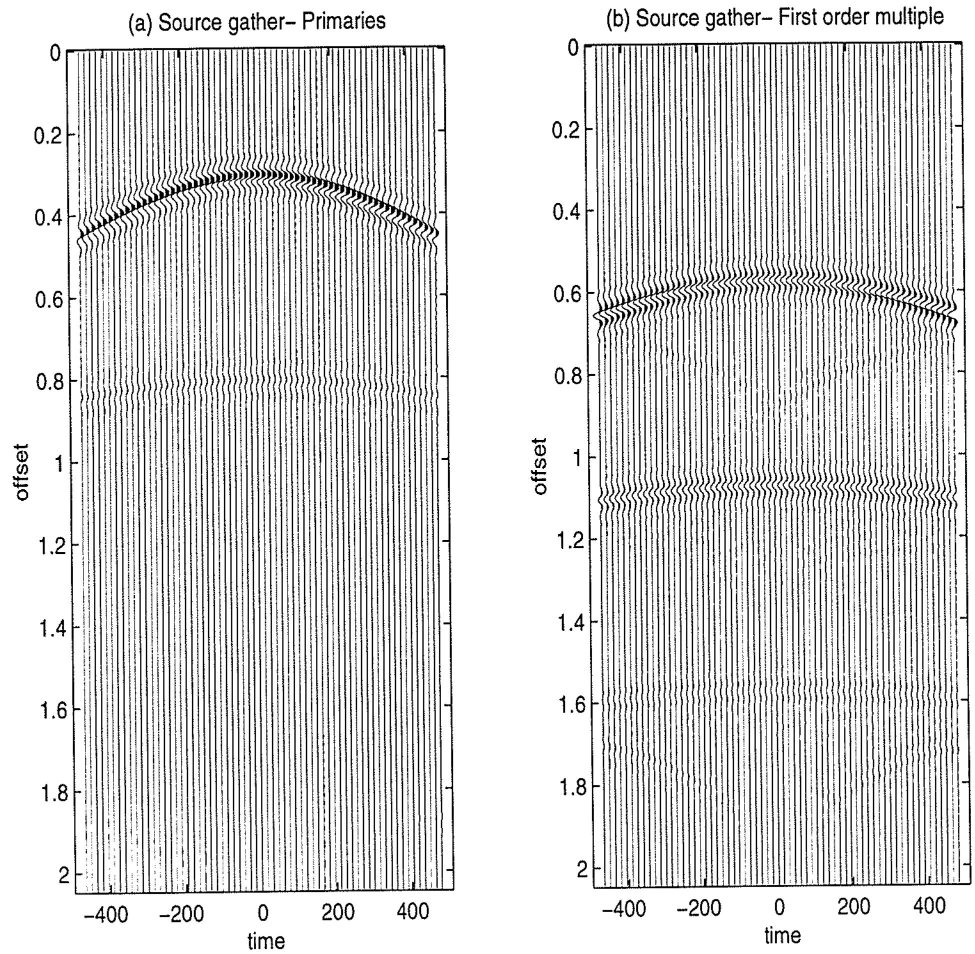

Figure 10: (a) Primaries (two layers and halfspace). (b) First order surface multiples.

P_{11} at $0.4s$

P_{12} at $0.5s$

P_{112} at $0.7s$

P_{122} at $0.8s$

P_{1122} at $1.0s$

P_{11222} at $1.3s$

P_{112222} at $1.6s$

where indexes represent the number of times that the wave has travelled through the layer

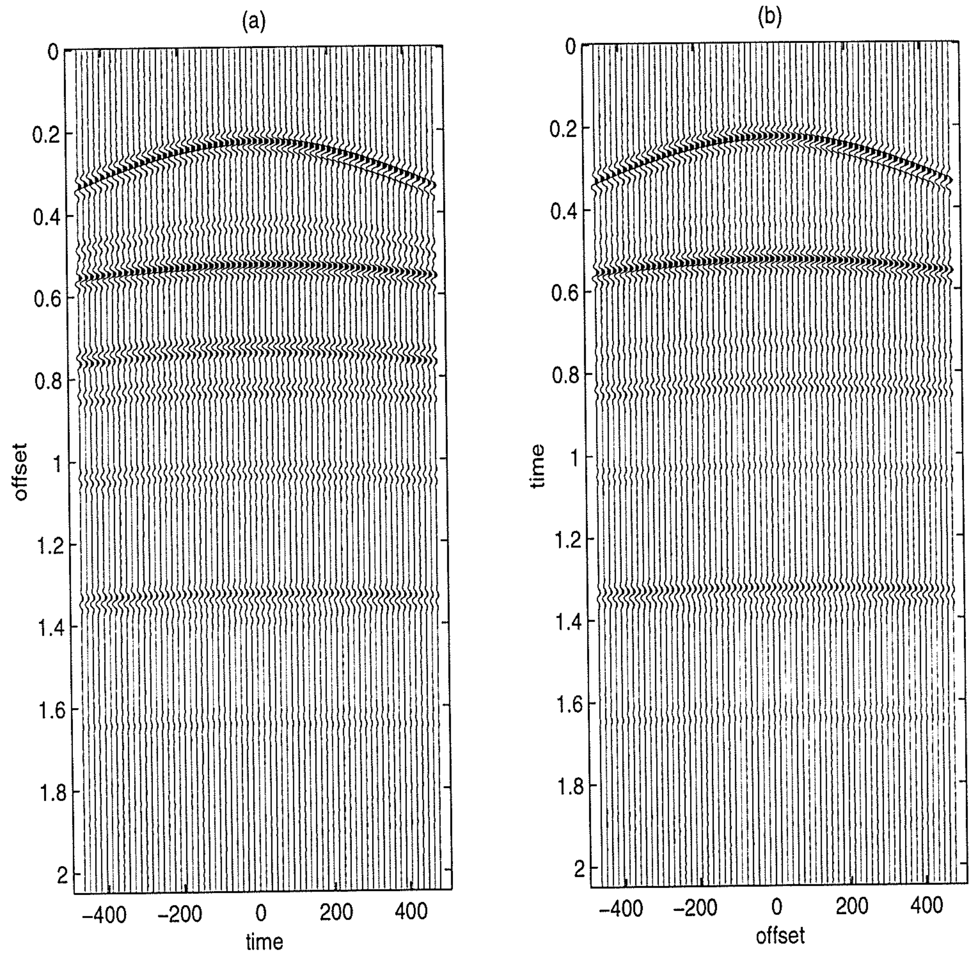

Figure 11: Surface multiple attenuation. (a) Primaries, first order surface multiples and internal multiples. (b) Surface multiples attenuated.

1 or 2. Figure 11b shows the wavefield after attenuation of the first order surface multiples. The following reflections have been attenuated,

P_{11} at $0.4s$

P_{112} at $0.7s$

P_{1122} at $1.0s$.

Primaries and internal multiples have not been altered

3.4 Summary

The method of wavefield extrapolation to calculate the response of the earth is explained and implemented. The forward model contains all shot and receiver gathers, and multiples can be included in the model. The model can be calculated in the frequency-space domain or the frequency wavenumber domain. Another version in the time space domain is possible but not considered here.

Essential to the formulation is the concept of the propagator, that can be evaluated for the earth model either in frequency-space or frequency-wavenumber. This operator extrapolates the wavefield to produce the surface data with all the reflections from the subsurface. The model is started at the bottom halfspace and all the internal reflections are added to the wavefield in its way to the surface. The source is incorporated at the end. Examples of forward and inverse models are shown. Finally the wavefields computed are used to exemplify the method of surface multiple attenuation.

Still some modifications are required in the codes to produce better results and, in particular, to apply the method to real data, but the presented examples show how all source and receiver gathers can be modeled, and multiples predicted and attenuated from the data.

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