Multiple removal based on wavefield extrapolation

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1 Introduction

Either forward or inverse modeling is required to deal with the properties of the subsurface that determine the reflected waves. Inverse modeling becomes very complicated, particularly when there are problems of instability and non-uniqueness. Forward modeling is simpler, but even so it becomes complicated for a general situation. In this section I will present the basic theory and simple examples concerning forward and inverse modeling following closely the work of Berkhout (1985). At the end the theory of multiple attenuation based on wavefield extrapolation will be considered.

The data represent upgoing reflected waves, related to downgoing source waves. Hence, a wave separation of the measured seismic data must be previously applied. This method is called one way approach because the downgoing and upgoing fields are computed separately. Another approach is the two way technique in that the total response (primaries and multiples) is computed by extrapolating simultaneously the continuous wavefield components P and $\frac{1}{\rho}\frac{\partial P}{\partial z}$.

2 Theory

Wavefield extrapolation is based on the Kirchhoff integral, which comes from substituting the wave equation into Green's second theorem

$$\int_V \left[F\nabla^2 G - G\nabla^2 F\right] dV = \oint_S \left[F\nabla G - G\nabla F\right].\hat{\mathbf{n}}\; dS. \tag{1}$$

Suppose we have a closed, source free surface S. We choose F as the pressure field which is generated by sources outside S

$$F = P(x, y, z, \omega) \tag{2}$$

where P satisfies

$$\nabla^2 P + k^2 P = 0. \tag{3}$$

G is chosen as the Fourier Transformed pressure for a compressional wavefield which is generated by a monopole in a point A inside S (Green's function)

$$G = \frac{\exp(-jkr)}{r} \tag{4}$$

with

$$r = \sqrt{(x-x_a)^2 + (y-y_a)^2 + (z-z_a)^2}. \tag{5}$$

G satisfies

$$\nabla^2 G + k^2 G = -4\pi\delta(x-x_a)\delta(y-y_a)\delta(z-z_a) \tag{6}$$

and Green's theorem becomes

$$\oint_S [P\nabla G - G\nabla P]\,.\vec{n}\,dS = -4\pi \int_V P\delta(x-x_a)\delta(y-y_a)\delta(z-z_a)dV \tag{7}$$

or using the shift property

$$\oint_S [P\nabla G - G\nabla P]\,.\hat{\mathbf{n}}\;dS = -4\pi P_A. \tag{8}$$

where P_A is the wavefield in x_a, y_a. The motion equation states the relation between acceleration and the pressure gradient

$$\frac{\partial P}{\partial n} = -\rho_0 \frac{\partial V_n}{\partial t}. \tag{9}$$

As usual we consider $V(x,y,z,t) = f(x,y,z)\exp(-j\omega t)$

$$\frac{\partial P}{\partial n} = \rho_0(j\omega\rho_0 V_n). \tag{10}$$

By substituting

$$\oint_S \left[P\frac{\partial G}{\partial n} + (j\omega\rho_0 V_n)G\right].\hat{\mathbf{n}}\;dS = -4\pi P_A \tag{11}$$

Substituting G, and $\frac{\partial G}{\partial n}$

$$P_A = \frac{1}{4\pi}\oint_S \left[P\frac{\partial\left(\frac{\exp(-jkr)}{r}\right)}{\partial n} + (j\omega\rho_0 V_n)\frac{\exp(-jkr)}{r}\right].\hat{\mathbf{n}}\;dS \tag{12}$$

This expression tell us how to compute P_A at any point in a source free halfspace due to sources in the other halfspace. Using the Green's function in Eq. (4), the normal derivative results in

$$\frac{\partial G}{\partial n} = \frac{1+jkr}{r^2}\exp(-jkr)\cos\phi \tag{13}$$

$$\cos\phi = \frac{\partial r}{\partial n} \tag{14}$$

Equation (12) is the Kirchhoff's integral for a homogeneous medium. This integral integral states that any pressure field may be synthesized by means of a monopole and a dipole distribution on a closed surface S. The strength of each monopole is given by the normal component of the velocity in S, the strength of each dipole is given by the pressure in S.

Kirchhoff's integral is not very useful in practice because we need to know the pressure and particle velocity data on a closed surface. However, from the Kirchoff integral it is possible to derive the Rayleigh integrals, which are very useful for seismic applications.

Let us choose for closed surface S the plane $z = 0$ and a hemisphere in the top half space. Letting the radius of the hemisphere becomes infinitely large, we can reduce the surface integral to the plane $z = 0$. Eq. (4) for Green's function was a particular function chosen to satisfy the Green's second theorem but it also could be chosen with a constant H and still satisfy Green's theorem

$$G = \frac{\exp(-jkr)}{r} + H. \tag{15}$$

If H is such that

$$\frac{\partial G}{\partial n} = \frac{1 + jkr}{r^2} \exp(-jkr) \cos\phi = 0 \tag{16}$$

on $z = 0$, the same previous derivation for P_A produces the so called Rayleigh integral of type I:

$$P_A = \frac{j\omega\rho_0}{2\pi} \int_{Lx} \int_{Ly} V_n \frac{\exp(-jkr)}{r} dxdy \tag{17}$$

and it represents the pressure in S due to a monopole source on $z = 0$.

By a similar argument we can choose H such that $G = 0$ for $z = 0$. This produces the Rayleigh integral of the second kind. The Rayleigh II integral states that any pressure field can be synthesized by a dipole distribution on the plane $z = 0$.

$$P(x_A, y_A, z_{i-1}, \omega) = \frac{1}{2\pi} \int_{Lx} \int_{Ly} P(x, y, z_i, \omega) \frac{1 + jkr}{r^2} \exp(-jkr) \cos\phi dxdy \tag{18}$$

where $\cos\phi = \frac{\partial r}{\partial n} = \frac{z}{r}$

or its 2D version

$$P(x_A, z_{i-1}, \omega) = -\frac{jk}{2} \int_{Lx} P(x, z_i, \omega) H_1^{(2)}(k\Delta r) \cos\phi dx \tag{19}$$

where $H_1^{(2)}$ is the first order Hankel function of the second class

$$H_1^{(2)} = \frac{1}{j\pi}\int_{-\infty}^{0} \frac{\exp(z/2)(t-1/t)}{t^2}dt \tag{20}$$

To illustrate the wavefield extrapolation principles I will use the 2D version of the Rayleigh integral of type II, i.e., equation (19). Wavefield extrapolation can be performed in space-time, space-frequency or wavenumber-frequency. I will describe here space-frequency and wavenumber-frequency approaches only.

Defining

$$W(x_A - x, \Delta z, \omega) = -\frac{jk}{2}\cos\phi H_1^{(2)}(kr) \tag{21}$$

we can write

$$P(x_A, z_{i-1}, \omega) = \frac{1}{2\pi}\int_{Lx} W(x_A - x, \Delta z, \omega) P(x, z_i, \omega) dx \tag{22}$$

or

$$P(x, z_{i-1}, \omega) = W(x, \Delta z, \omega) * P(x, z_i, \omega) \tag{23}$$

For the 3D case

$$W(x_A - x, y_A - y, \Delta z, \omega) = \frac{jk}{2\pi}\left[\frac{1 + jkr}{jkr}\cos\phi\right]\frac{\exp(-jkr)}{r}. \tag{24}$$

and

$$P(x, y, z_{i-1}, \omega) = W(x, y, \Delta z, \omega) * P(x, y, z_i, \omega) \tag{25}$$

Hence forward extrapolation in the space-frequency domain can be formulated in terms of convolution along the spatial axes, x and y. If we consider a dipole at $z = z_i$, then the response at depth level $z = z_{i-1}$ is given by $W(x_r - x_s, y_r - y_s, \Delta z, \omega)$, so that W is called the spatial impulse response or the spatial wavelet for the temporal frequency ω. If the velocity varies laterally, the spatial wavelet $W(x_r - x_s, y_r - y_s, \Delta z, \omega)$ becomes space-variant, that is $W(x_s, x_r - x_s, y_r - y_s, \Delta z, \omega)$, and the convolution is valid only if an average value of the velocity can be used within the operator length. Generally, for space-variant discrete situations and a finite operator length, a matrix formulation is used.

If we consider the situation without lateral variations in thickness or velocity, then the function W does not change along the spatial coordinates x and y so that the convolution can be carried out by means of a multiplication in the ω-k_x domain

$$\widetilde{P}_{i-1}(k_x, k_y, z_{i-1}, \omega) = \widetilde{W}(k_x, k_y, \Delta z, \omega)\widetilde{P}_i(k_x, k_y, z_i, \omega) \tag{26}$$

where

$$\widetilde{W}(k_x, k_y, \Delta z, \omega) = \mathcal{F}[\frac{j|z|}{2\pi}(\frac{1+jkr}{jkr^3})\exp(-jkr)] = \exp(-j\sqrt{\frac{\omega^2}{c^2} - (k_x^2 + k_y^2)}\ \Delta z) \tag{27}$$

$$\widetilde{W}(k_x, k_y, \Delta z, \omega) = \exp(-jk_z\Delta z) \tag{28}$$

F stands for Fourier transform and k_z is the vertical component of the wavenumber.

This result can be also obtained from the Helmholtz equation. Substituting a solution like

$$p = p(x, y, z)\exp(-i\omega t) \tag{29}$$

into the wave equation we obtain

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0. \tag{30}$$

Fourier transformation with respect to x and y produces the one-dimensional Helmholtz equation;

$$\frac{\partial^2 P}{\partial z^2} + (k^2 - k_x^2 - k_y^2)P = 0 \tag{31}$$

whose solution is

$$P(k_x, k_y, z, \omega) = A(k_x, k_y, \omega)\exp(\pm jk_z|z - z_i|) \tag{32}$$

By taking $z \longrightarrow z_i$ the integration constant $A_i = P(k_x, k_y, z_i, \omega)$ and

$$P(k_x, k_y, z, \omega) = P(k_x, k_y, z_i, \omega)\exp(\pm jk_z|z - z_i|) \tag{33}$$

With the propagator $W(x, y, z_i, z_{i-1}, \omega)$ it is possible to perform the forward extrapolation of P from z_{i-1} to z_i. This operator can be applied recursively to go from any z_i to any z_j

where $z_j > z_i$. In the same way an inverse extrapolator can be defined to go from z_i to z_{i-1} or working recursively from any z_j to z_i where $z_j < z_i$.

Now, if we have a downward propagating source $S^+(x, y, \omega)$ at the surface and a series of reflectivities $R(x, y, z_m, \omega)$,with $m = 1, N$, the wavefield $P(x, y, z_0, t)$ can be obtained as

$$P(x, y, z_0, t) = \mathcal{F}^{-1}[D(z_{0,}\omega) * [\sum_{m=1}^{N} W^-(z_0, z_m, \omega) * R(z_m, \omega) * W^+(z_m, z_0, \omega)] * S^+(z_0, \omega)] \quad (34)$$

where $\mathcal{F}^{-1}[.]$ is the inverse Fourier Transform, and the notation has been simplified making implicit that all terms are x, y dependent, $D(x, y, z_{0,}\omega)$ accounts for the properties of the receivers and, in the simplest case $D(x, y, z_{0,}\omega) = I$.

If we reverse one of the operators, set it as block tridiagonal matrix and set the other operator as columns of a second matrix, the convolutions can be carried out by means of matrix multiplication. To do this a matrix $P(\omega)$ is built up for each frequency setting the shot gathers as columns. In this way the rows contain the receiver gathers. To visualize this, consider a 2D data set with different shots, each one with its corresponding shot gather. Setting every shot gather behind the previous one, a 3 dimensional matrix is constructed as $\mathbf{P}(t, x_r, x_s)$ where t is the time, x_r represents the coordinates of the receivers, and x_s the coordinates of the sources. Fourier transforming produces $\mathbf{P}(\omega, x_r, x_s)$. An index permutation gives $\mathbf{P}(x_r, x_s, \omega)$. With these matrices and the shifted operator $\mathbf{W}(x, z_0, z_m\omega)$ set as

$$\begin{matrix} \overrightarrow{\mathbf{W}}(-x, z_0, z_m\omega_i) & \overrightarrow{0} & \dots & \overrightarrow{0} \\ 0 & \overrightarrow{\mathbf{W}}(-x+1, z_0, z_m\omega_i) & \dots & \overrightarrow{0} \\ \dots & \dots & \dots & \dots \\ \overrightarrow{0} & \overrightarrow{0} & .. & \overrightarrow{\mathbf{W}}(-x+n, z_0, z_m\omega_i) \end{matrix} \quad (35)$$

where all the elements are vectors, the convolution can be performed as a matrix multiplication,

$$\overrightarrow{\mathbf{W}}(x, z_0, z_m, \omega_i) * \overrightarrow{\mathbf{P}}(x, z_m, \omega_i) = \mathbf{W}(x, z_0, z_m, \omega_i)\overrightarrow{\mathbf{P}}^{\mathbf{T}}(x, z_m, \omega_i). \quad (36)$$

Hence it is possible to compute the response of the earth, frequency by frequency, with all the sources.

$$\mathbf{P}(z_0, \omega) = \mathbf{D}(z_{0,}\omega)[\sum_{m=1}^{N} \mathbf{W}^-(z_0, z_m, \omega)\mathbf{R}(z_m, \omega)\mathbf{W}^+(z_m, z_0, \omega)]\mathbf{S}^+(z_0, \omega). \quad (37)$$

Here all the terms are x, y dependent and

$$\mathbf{W}^+(z_m, z_0, \omega) = \mathbf{W}^+(x, y, z_m, z_0, \omega)$$

is the downward propagator from z_0 to z_m,

$$\mathbf{W}^-(z_m, z_0, \omega) = \mathbf{W}^-(x, y, z_m, z_0, \omega)$$

is the upward propagator from z_m to z_0,

$$\mathbf{P}(z_i, \omega) = \mathbf{P}(x, y, z_i, \omega)$$

is the wavefield at $z = z_i$. In the simplest 1D case (no x, y variation in the earth properties) with sources along x it is possible to work in the $k_x - \omega$ domain

$$P(z_0, k_x, \omega) = D(z_0, k_x, \omega).[\sum_m W^-(z_0, z_m, k_x, \omega) R(z_m, k_x, \omega) W^+(z_m, z_0, k_x, \omega)] S(z_0, k_x, \omega) \tag{38}$$

where all matrix multiplications have been replaced by scalar multiplication.

Let us simplify the problem taking $\mathbf{D} = \mathbf{I}$, that is one receiver at every x position and calling

$$\mathbf{T}(z_0) = \sum_m \mathbf{W}^-(x, y, z_0, z_m, \omega) \mathbf{R}(x, y, z_m, \omega) \mathbf{W}^+(x, y, z_m, z_0, \omega), \tag{39}$$

$\mathbf{T}(z_0)$ is an operator which takes the downward wavefield P^+ at level z_0, downward propagates it until $z = z_m$, computes the reflected upward wavefield P^- at $z = z_m$, and propagates it upward until $z = z_0$ and finally adds all the wave fields coming from all layers m. In the absence of multiples the wavefield at $z = z_0$ will be

$$\mathbf{P}^-(z_0) = \mathbf{T}(z_0)\mathbf{S}^+ \tag{40}$$

where $\mathbf{S}^+$ is the downward field from the source.

Multiples can be included in this formulation because these are generated by feed back of previously generated waves. An expression which includes surface related multiples, i.e., all multiple reflections that have been reflected at least once from the free surface can be obtained as follows. If there is a surface with reflection coefficient r_0 the upward wavefield $\mathbf{P}^-(z_0)$ will produce a downward field

$$\mathbf{P}^+(z_0) = -r_0 \mathbf{P}^-(z_0). \tag{41}$$

The total downward field will be

$$\mathbf{P}^+_{\mathbf{inc}}(z_0) = \mathbf{S}^+ - r_0 \mathbf{P}^-(z_0) \tag{42}$$

and the response of the earth will be

$$\mathbf{P}^-(z_0) = \mathbf{T}(z_0)\mathbf{P}^+_{\mathbf{inc}}(z_0) \tag{43}$$

or

$$\mathbf{P}^{-}(z_0) = \mathbf{T}(z_0)[\mathbf{S}^{+} - r_0\mathbf{P}^{-}(z_0)] \tag{44}$$

which is a recursive filter, whose present output depends on past outputs.

$$\mathbf{P}^{-}(z_0) + {r_0}^{+}\mathbf{T}(z_0)\mathbf{P}^{-}(z_0) = \mathbf{T}(z_0)\mathbf{S}^{+} \tag{45}$$

Solving for $\mathbf{P}^{-}(z_0)$ the formulation

$$\mathbf{P}^{-}(z_0) = [\mathbf{I} + r_0\mathbf{T}(z_0)]^{-1}\mathbf{T}(z_0)\mathbf{S}^{+} \tag{46}$$

generates all the free surface multiples. Expanding

$$[\mathbf{I} + r_0\mathbf{T}(z_0)]^{-1} = \sum_{m=0}^{\infty}(-r_0)^{\mathbf{n}}\mathbf{T}^{\mathbf{n}}(z_0) \tag{47}$$

$$\mathbf{P}^{-}(z_0) = \mathbf{P}^{-}(z_0) + \sum_{m=1}^{\infty}(-r_0)^{n}\mathbf{T}^{\mathbf{n}}(z_0)\mathbf{S} \tag{48}$$

where taking more terms produces higher order surface multiples. To include all possible multiples, free surface as well as internal, we have to extend the recursive primary modeling scheme further by adding, during each upward continuation step, all multiples related to the current surface. For example, if we have arrived at the level z_{m+1} with an upward travelling reflected wavefield $\mathbf{P}^{-}(z_{m+1})$ which includes all multiples, then

$$\mathbf{P}^{-}(z_m) = \mathbf{W}^{-}(z_m, z_{m+1})[\mathbf{R}^{+}(z_{m+1}) + \mathbf{P}^{-}(z_{m+1})]\mathbf{W}^{+}(z_{m+1}, z_m) \tag{49}$$

where $\mathbf{P}^{-}(z_m)$ represents the total upward travelling response from depth level $z > z_m$ assuming zero reflectivity at $z = z_m$. To include all multiples associated with $z = z_m$ we need another step to include the feedback of $\mathbf{P}^{-}(z_m)\ \mathbf{R}^{-}(z_m)$.

$$\mathbf{P}^{-}_{\mathbf{tot}}(z_m) = [\mathbf{I} - \mathbf{P}^{-}(\mathbf{z_m}).\mathbf{R}^{-}(z_m)]^{-1}\mathbf{P}^{-}(z_m) \tag{50}$$

or to reduce instability,

$$\mathbf{P}^{-}_{\mathbf{tot}}(z_m) = \mathbf{P}^{-}(z_m) + \sum_{n=1}^{\infty}[\mathbf{R}^{-}(\mathbf{z_m}).\mathbf{P}^{-}(z_m)]^{n}\mathbf{P}^{-}(z_m) \tag{51}$$

where only some of the infinity terms are calculated. In conclusion, we start at maximum depth and continue up to the surface according to the previous expressions (49) and (50). When we arrive at the surface, we have created the total response, i.e., primaries and all possible multiple reflections. In the last step the source matrix and the detector matrix can be included.